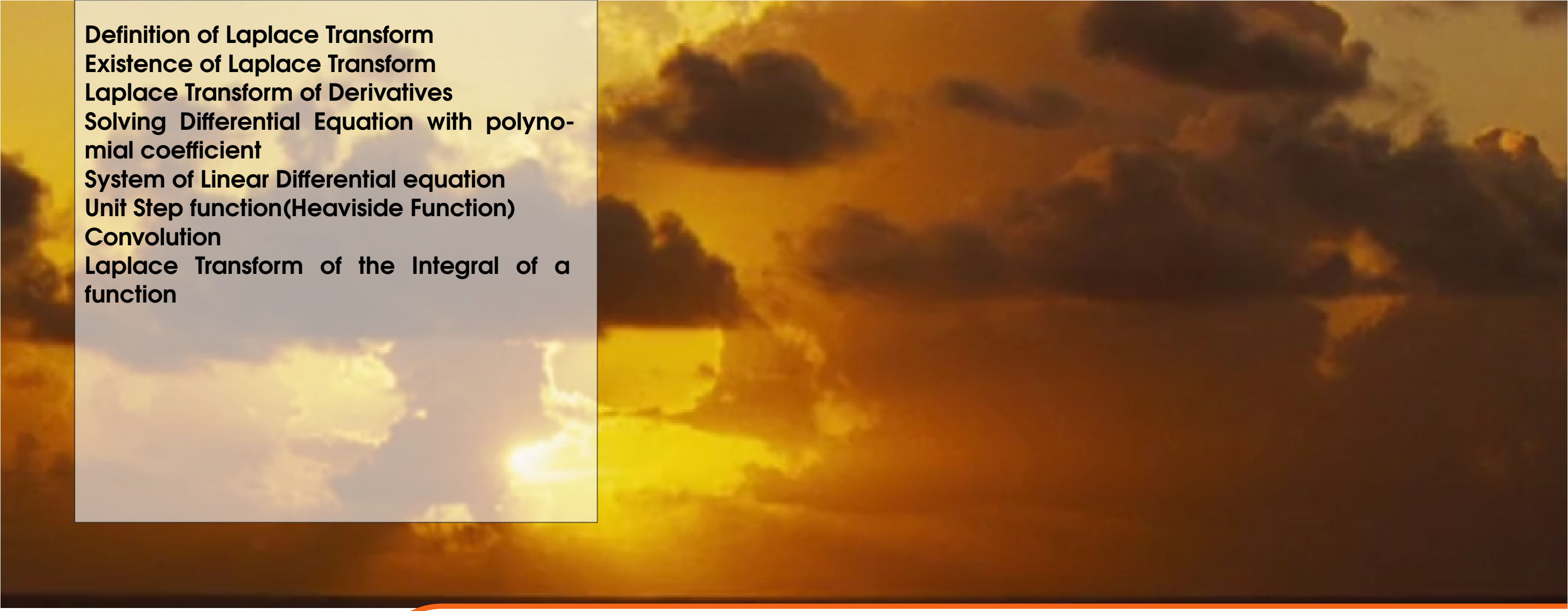


3 — Laplace Transform

3.1 Definition of Laplace Transform

Definition 3.1.1 The Laplace transform of a function $f(t)$, denoted by $F(s) = \mathscr{L}(f(t))$ is a function defined by

$$F(s) = \mathscr{L}(f(t)) = \int_0^\infty e^{-st} f(t)dt \tag{3.1}$$

for all s such that this integral converges.

Definition 3.1.2 The orginal function $f(t)$ in (3.1) is called the inverse transform or invers of $F(s)$ and will be denoted by $\mathscr{L}^{-1}(F(s))$

$$\text{i.e., } f(t) = \mathscr{L}^{-1}(F(s))$$

Example 3.1 Let $f(t) = 1$ when $t \geq 0$. Find $F(s)$

Solution: From (3.1) we obtain

$$\begin{aligned}
\mathscr{L}(f(t)) &= \mathscr{L}(1) = \int_0^\infty e^{-st}dt = \lim_{n\to\infty}\int_0^n e^{-st}dt = \lim_{n\to\infty}\left(\frac{-1}{s}e^{-st}\right)\Big|_0^n \\
&= \lim_{n\to\infty}\left(\frac{-1}{s}(e^{-sn}-1)\right) \\
&= \frac{1}{s}, \ (s>0)
\end{aligned}$$

Example 3.2 Let $f(t) = e^{at}$ when $t \geq 0$, where a is constant. Find $\mathscr{L}(f)$

Solution: By definition

$$\begin{aligned}
F(s) = \mathscr{L}(f(t)) &= \mathscr{L}(e^{at}) = \int_0^\infty e^{-st}e^{at}dt = \lim_{n\to\infty}\int_0^n e^{-(s-a)t}dt = \lim_{n\to\infty}\left(\frac{-1}{s-a}e^{-(s-a)t}\right)\Big|_0^n \\
&= \frac{1}{s-a}, \ (s>a)
\end{aligned}$$

Theorem 3.1.1 — Linearity of the Laplace Transform The Laplace transform is linear operation; that is, for any function $f(t)$ and $g(t)$ whose Laplace transforms exist and any constants a & b,

$$\mathscr{L}\{af(t)+bg(t)\} = a\mathscr{L}(f(t))+b\mathscr{L}(g(t))$$

Example 3.3 Let $f(t) = \sin\omega t$. Find $\mathscr{L}(f(t))$

Solution: Since $\mathscr{L}(e^{at}) = \dfrac{1}{s-a}$, set $a = i\omega$ with $i = \sqrt{-1}$

$$\Rightarrow \quad \mathscr{L}(e^{i\omega t}) = \frac{1}{s-i\omega} = \frac{s+i\omega}{(s-i\omega)(s+i\omega)} = \frac{s+i\omega}{s^2+\omega^2} = \frac{s}{s^2+\omega^2} + i\frac{\omega}{s^2+\omega^2}$$

Since $e^{i\omega t} = \cos\omega t + i\sin\omega t$ (Euler's Formule) and by theorem 3.1.1, we obtain

$$\begin{aligned} \mathscr{L}(e^{i\omega t}) &= \mathscr{L}(\cos\omega t + i\sin\omega t) \\ &= \mathscr{L}(\cos\omega t) + i\mathscr{L}(\sin\omega t) \end{aligned}$$

Equating the real and imaginary parts of these two equations, we get

$$\mathscr{L}(\cos\omega t) = \frac{s}{s^2+\omega^2} \text{ and } \mathscr{L}(\sin\omega t) = \frac{\omega}{s^2+\omega^2}$$

	$f(t)$	$\mathscr{L}(f(t)) = F(s)$	Domain
1	c (constant)	$\frac{c}{s}$	$s > 0$
2	t	$\frac{1}{s^2}$	$s > 0$
3	t^n	$\frac{n!}{s^{n+1}}$	$s > 0$
4	e^{kt}	$\frac{1}{s-k}$	$s > k$
5	$\sin kt$	$\frac{k}{s^2+k^2}$	$s > 0$
6	$\cos kt$	$\frac{s}{s^2+k^2}$	$s > 0$
7	$\cosh kt$	$\frac{s}{s^2-k^2}$	$s > k$
8	$\sinh kt$	$\frac{k}{s^2-k^2}$	$s > k$

Theorem 3.1.2 — (First Shifting Theorem) If $f(t)$ has the transform $F(s)$ (where $s > k$), then $e^{at}f(t)$ has the transform $F(s-a)$ (where $s-a > k$)

$$\begin{aligned} \text{i.e., } \mathscr{L}(e^{at}f(t)) &= F(s-a) \text{ or} \\ e^{at}f(t) &= \mathscr{L}^{-1}(F(s-a)) \end{aligned}$$

Example 3.4 Compute $\mathscr{L}(e^{2t}\cos 3t)$

Solution: Since $F(s) = \mathscr{L}(\cos 3t) = \frac{s}{s^2+9}$ and $a = 2$, we have $\mathscr{L}(e^{2t}\cos 3t) = \frac{s-2}{(s-2)^2+9}$

■ **Example 3.5** Find $\mathscr{L}^{-1}\left(\frac{1}{s-4} - \frac{6}{(s-4)^2}\right)$ ■

Solution: Since $\mathscr{L}^{-1}\left(e^{4t}\right) = \frac{1}{s-4}$ and $\mathscr{L}(t) = \frac{1}{s^2} \Rightarrow \mathscr{L}(te^{4t}) = \frac{1}{(s-4)^2}$

$$\therefore \quad \mathscr{L}^{-1}\left(\frac{1}{s-4} - \frac{6}{(s-4)^2}\right) = \mathscr{L}^{-1}\left(\frac{1}{s-4}\right) - 6\mathscr{L}^{-1}\left(\frac{1}{(s-4)^2}\right) = e^{4t} - 6te^{4t}$$

Exercise 3.1 Compute

1. $\mathscr{L}(e^{at}t^n)$
2. $\mathscr{L}^{-1}\left(\frac{1}{s^2+4s+13}\right)$
3. $\mathscr{L}^{-1}\left(\frac{2s+4}{s^2+4s+5}\right)$

■

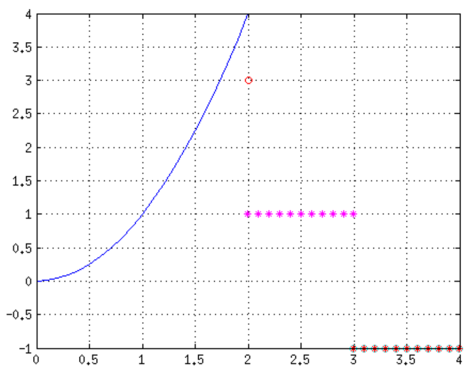
3.2 Existence of Laplace Transform

Definition 3.2.1 A function f has a jump discontinuous at a point t_0 if the function has different (finite) limits at t approaches t_0 from the left and from the right or if the two limits are equal but different from $f(t_0)$. Note that $f(t_0)$ may or may not be equal to either

$$\lim_{t \to t_0^+} f(t) \text{ or } \lim_{t \to t_0^-} f(t)$$

Definition 3.2.2 A function f defined on $(0, \infty)$ is picewise continuous if it is continuous on every finite interval $0 \le t \le \infty$, except possibly at finitely many points where it has jump discontinuties

■ **Example 3.6** Let $\begin{cases} t^2 & \text{for} \quad 0 \le t < 2 \\ 3 & \text{for} \quad t = 2 \\ 1 & \text{for} \quad 2 < t \le 2 \\ -1 & \text{for} \quad 3 < t \le 4 \end{cases}$ ■



Theorem 3.2.1 — Existence Theorem Let $f(t)$ be a function which is picewise continuous on every finite interval in the range $t \geq 0$ and satisfies

$$|f(t)| \leq Me^{kt}, \quad \forall t \geq 0 \tag{3.2}$$

and for some constant k and M. Then the Laplace transform of $f(t)$ exists for all $s > k$

Proof. Since f is piecewise continuous, $e^{-st}f(t)$ has a finite integral over any finite interval on $t \geq 0$, and

$$\begin{aligned} |\mathscr{L}(f(t))| = \left|\int_0^\infty e^{-st}f(t)dt\right| &\leq \int_0^\infty e^{-st}|f(t)|dt \\ &\leq \int_0^\infty Me^{-st}e^{kt}dt = M\int_0^\infty e^{-(s-k)t}dt \\ &= \frac{M}{s-k}, \quad s > k \end{aligned}$$

$\mathscr{L}(f(t))$ exists. (comparison theorem) ■

3.3 Laplace Transform of Derivatives

Theorem 3.3.1 Suppose that $f(t)$ is contiuous for all $t \geq 0$, satisfies the condition

$$|f(t)| \leq Me^{kt}$$

for some k and M, and has a derivative $f'(t)$ that is piecewise continuous on every finite interval in the range $t \geq 0$. Then the Laplace transform of the derivative $f'(t)$ exists when $s > k$ and

$$\mathscr{L}(f'(t)) = s\mathscr{L}(f(t)) - f(0) \tag{3.3}$$

Proof. Suppose $f'(t)$ is continuous for all $t \geq 0$. Integrating by parts

$$\begin{aligned} \mathscr{L}(f'(t)) &= \int_0^\infty e^{-st}f'(t)dt \;=\; e^{-st}f(t)|_0^\infty + \int_0^\infty se^{-st}f(t)dt \\ &= 0 - f(0) + s\int_0^\infty e^{-st}f(t)dt = -f(0) + s\mathscr{L}(f(t)) \\ \therefore\; \mathscr{L}(f'(t)) &= s\mathscr{L}(f(t)) - f(0) \end{aligned}$$

■

Theorem 3.3.2 — Laplace transform of the derivative of any order n Let $f(t)$ and its derivatives $f'(t),\ f''(t),\ \ldots,\ f^{(n-1)}(t)$ be continuous functions for all $t \geq 0$, satisfies the condition

$$|f(t)| \leq Me^{kt}$$

for some k and M, and let the derivative $f^{(n)}(t)$ be piecewise continuous on every finite interval in the range $t \geq 0$. Then the Laplace transform of the derivative $f^{(n)}(t)$ exists when $s > k$ and is given by

$$\mathscr{L}(f^{(n)}(t)) = s^n\mathscr{L}(f(t)) - s^{n-1}f(0) - s^{n-2}f'(0) - \ldots, -f^{(n-1)(0)} \tag{3.4}$$

For $n = 2$, $\mathscr{L}(f''(t)) = s^2\mathscr{L}(f(t)) - sf(0) - f'(0)$

Example 3.7 Let $f(t) = \cos^2 t$. Find $\mathscr{L}(f(t))$

Solution: We have $f(0) = 1,\ f'(t) = -2\cos t \sin t = -\sin 2t$

$$
\begin{aligned}
\mathscr{L}(f'(t)) &= \mathscr{L}(-\sin 2t) = s\mathscr{L}(f(t)) - f(0) \\
\Rightarrow \frac{-2}{s^2+4} &= s\mathscr{L}(\cos^2 t) - 1 \ \Rightarrow\ s\mathscr{L}(\cos^2 t) = 1 - \frac{2}{s^2+4} \\
\therefore\ \mathscr{L}(\cos^2 t) &= \frac{s^2+2}{s(s^2+4)}
\end{aligned}
$$

Example 3.8 Let $f(t) = t\sin \omega t$. Find $\mathscr{L}(f(t))$

Solution: We have $f(0) = 0,\ f'(t) = \sin \omega t + t\omega \cos \omega t \Rightarrow f'(0) = 0$
$f''(t) = 2\omega \cos \omega t - t\omega^2 \sin \omega t$

$$
\begin{aligned}
& \mathscr{L}(f''(t)) = s^2\mathscr{L}(f(t)) - sf(0) - f'(0) \\
\Rightarrow\ & \mathscr{L}(2\omega \cos \omega t - t\omega^2 \sin \omega t) = s^2\mathscr{L}(t\sin \omega t) - s.0 - 0 \\
\Rightarrow\ & 2\omega\mathscr{L}(\cos \omega t) - \omega^2\mathscr{L}(t\sin \omega t) = s^2\mathscr{L}(t\sin \omega t) \\
\Rightarrow\ & \frac{2\omega s}{s^2+\omega^2} = (s^2+\omega^2)\mathscr{L}(t\sin \omega t) \\
\Rightarrow\ & \mathscr{L}(t\sin \omega t) = \frac{2\omega s}{(s^2+\omega^2)^2}
\end{aligned}
$$

Example 3.9 Solve the IVP
a) $y'' - 4y = 0,\ \ y(0) = 1,\ y'(0) = 2$ b) $y'' - 3y' + 2y = 4t - 6,\ \ y(0) = 1,\ y'(0) = 3$

Solution: a) Taking Laplace transform both sides and using differentiation property, we have

$$
\begin{aligned}
\mathscr{L}(y'' - 4y) &= \mathscr{L}(y'') - 4\mathscr{L}(y) = 0 \ \Rightarrow\ s^2\mathscr{L}(y) - sy(0) - y'(0) - 4\mathscr{L}(y) = 0 \\
&\Rightarrow\ (s^2-4)\mathscr{L}(y) - s - 2 = 0 \\
&\Rightarrow\ \mathscr{L}(y) = \frac{s+2}{s^2-4} = \frac{1}{s-2} \\
&\Rightarrow\ y = \mathscr{L}^{-1}\left(\frac{1}{s-2}\right) = e^{2t}
\end{aligned}
$$

$\therefore\ y(t) = e^{2t}$

b) $y'' - 3y' + 2y = 4t - 6,\ \ y(0) = 1,\ y'(0) = 3$ Taking Laplace transform both sides

$$
\begin{aligned}
& \mathscr{L}(y'' - 3y' + 2y) = \mathscr{L}(4t - 6) \\
\Rightarrow\ & \mathscr{L}(y'') - 3\mathscr{L}(y') + 2\mathscr{L}(y) = 4\mathscr{L}(t) - \mathscr{L}(6) \\
\Rightarrow\ & s^2\mathscr{L}(y) - sy(0) - y'(0) - 3s\mathscr{L}(y) + 3y(0) + 2\mathscr{L}(y) = \frac{4}{s^2} - \frac{6}{s}
\end{aligned}
$$

$$\Rightarrow \quad (s^2-3s+2)\mathscr{L}(y)-s=\frac{4-6s}{s^2}$$
$$\Rightarrow \quad (s-2)(s-1)\mathscr{L}(y)=\frac{4-6s}{s^2}+s=\frac{4-6s+s^3}{s^2}$$
$$\Rightarrow \quad \mathscr{L}(y)=\frac{(s-2)(s^2+2s-2)}{s^2(s-2)(s-1)}=\frac{s^2+2s-2}{s^2(s-1)}$$
$$\Rightarrow \quad \mathscr{L}(y)=\frac{s^2}{s^2(s-1)}+\frac{2(s-1)}{s^2(s-1)}=\frac{1}{s-1}+\frac{2}{s^2}$$
$$\Rightarrow \quad y(t)=\mathscr{L}^{-1}\left(\frac{1}{s-1}\right)+2\mathscr{L}^{-1}\left(\frac{1}{s^2}\right)$$

$\therefore \; y(t)=e^t+2t$

Exercise 3.2 Solve the IVP

1. $y''+4y=0, \; y(0)=1, \; y'(0)=2$
2. $y''-5y'+6y=e^{-t}, \; y(0)=0, \; y'(0)=2$
3. $y'+4y=\cos t, \; y(0)=0$
4. $y''+4y'+3y=e^t, \; y(0)=1, \; y'(0)=2$

■

3.4 Solving Differential Equation with polynomial coefficient

Theorem 3.4.1 Let $\mathscr{L}(f(t))=F(s)$ for $s>a$, and suppose that F is differentiable. Then

$$\mathscr{L}(tf(t))=-\frac{d}{ds}\mathscr{L}(f(t)) \tag{3.5}$$

Proof. $F(s)=\int_0^\infty e^{-st}f(t)dt$

$$\frac{d}{ds}F(s)=\frac{d}{ds}\int_0^\infty e^{-st}f(t)dt=\int_0^\infty \frac{d}{ds}e^{-st}f(t)dt=\int_0^\infty -te^{-st}f(t)dt=-\mathscr{L}(tf(t))$$

■

Ingeneral,

$$\mathscr{L}(t^nf(t))=(-1)^n\frac{d^n}{ds^n}F(s) \tag{3.6}$$

■ **Example 3.10** Find $\mathscr{L}(e^{-t}t\sin 2t)$ ■

Solution:

$$\begin{aligned}
\mathscr{L}(e^{-t}t\sin 2t) &= -\frac{d}{ds}\mathscr{L}(e^{-t}\sin 2t)\\
&= -\frac{d}{ds}\left(\frac{2}{(s+1)^2+4}\right)\\
&= \frac{4(s+1)}{((s+1)^2+4)^2}
\end{aligned}$$

Example 3.11 Solve the equation with variable coefficients

$$ty'' - ty' - y = 0, \ \ y(0) = 0, \ y'(0) = 3$$

Solution: Let $Y(s) = \mathscr{L}(y(t))$

$$\begin{aligned}
\mathscr{L}(ty''(t)) &= -\frac{d}{ds}\mathscr{L}(y'') = -\frac{d}{ds}\left(s^2\mathscr{L}(y) - sy(0) - y'(0)\right) \\
&= -\frac{d}{ds}\mathscr{L}\left(s^2Y(s) - s.0 - 3\right) \\
&= -(2sY(s) + s^2Y'(s))
\end{aligned}$$

$$\begin{aligned}
\mathscr{L}(ty'(t)) &= -\frac{d}{ds}\mathscr{L}(y') = -\frac{d}{ds}(s\mathscr{L}(y) - y(0)) \\
&= -\frac{d}{ds}(sY(s)) = -(Y(s) + sY'(s))
\end{aligned}$$

Thus, $ty'' - ty' - y = 0, \ \ y(0) = 0$. Taking both sides Laplace transform.

$$\begin{aligned}
&\Rightarrow \ \mathscr{L}(ty'' - ty' - y) = 0 \ \Rightarrow \ \mathscr{L}(ty'') - \mathscr{L}(ty) - \mathscr{L}(y) = 0 \\
&\Rightarrow \ -(Y(s) + sY'(s)) + Y(s) + sY'(s) - Y(s) = 0 \\
&\Rightarrow \ (s - s^2)Y'(s) - 2sY(s) = 0 \ \Rightarrow \ s(s-1)Y' = 2sY \\
&\Rightarrow \ Y' = \frac{2}{1-s}Y \ \ \Rightarrow \ \ \frac{dY}{ds} = \frac{2}{1-s}Y \\
&\Rightarrow \ \frac{dY}{Y} = \frac{2}{1-s}ds \\
&\Rightarrow \ \ln Y = -2\ln(s-1) + \ln c \\
&\Rightarrow \ Y = \frac{c}{(s-1)^2} \ \ \Rightarrow \ \ \mathscr{L}(y) = \frac{c}{(s-1)^2} \\
&\Rightarrow \ y(t) = \mathscr{L}^{-1}\left(\frac{c}{(s-1)^2}\right) = cte^t
\end{aligned}$$

$y(0) = 0$ To find c, $y'(t) = ce^t + cte^t \Rightarrow y'(0) = c = 3$

$$\therefore \ \ y(t) = 3te^t$$

Exercise 3.3 Find

1. $\mathscr{L}(te^{-t}\sin 4t)$
2. $\mathscr{L}(t^2e^{3t}\cos 2t)$

Exercise 3.4 Solve the IVP

1. $ty'' - y' = 2t^2, \ \ y(0) = 0$
2. $ty'' + (4t-2)y' - 4y = 0, \ \ y(0) = 1$
3. $2y'' + ty' - 2y = 10, \ \ y(0) = y'(0) = 0$

3.5 System of Linear Differential equation

Consider

$$\begin{aligned} \frac{dx}{dt} &= a_{11}(t)x(t) + a_{12}(t)y(t) + f(t) \\ \frac{dy}{dt} &= a_{21}(t)x(t) + a_{22}(t)x(t) + g(t) \end{aligned} \tag{3.7}$$

with initial conditions $x(0) = x_0$ and $y(0) = y_0$
By taking Laplace transform both equations in system (3.7) we can find the solutions of the system.

Example 3.12 Consider the system of initial value equation

$$\begin{aligned} x' + y &= e^{2t} \\ x + y' &= 0 \end{aligned}$$

s.t $x(0) = 0,\ y(0) = 0$

Exercise 3.5 Solve system of differential equation

$$a. \quad \begin{aligned} x' + y &= 2\cos t \\ x + y' &= 0 \end{aligned}$$

s.t $x(0) = 0,\ y(0) = 1$

$$b. \begin{aligned} y_1'' &= y_1 + 3y_2 \\ y_2'' &= 4y_1 - 4e^t \end{aligned}$$

s.t $y_1(0) = 2,\ y_1'(0) = 3,\ y_2(0) = 1, y_2(0) = 2$

$$c \quad \begin{aligned} y_1' &= -y_2 \\ y_2' &= y_1,\ y_1(0) = 1,\ y_2(0) = 0 \end{aligned}$$

3.6 Unit Step function(Heaviside Function)

A unit step function is defined by

$$U(t-a) = U_a(t) = \begin{cases} 0 & \text{if} \quad t < a \\ & \\ 1 & \text{if} \quad t \geq a \end{cases} \qquad a \geq 0$$

A general piecewise-defined function of the type

$$f(t) = \begin{cases} g(t) & \text{if} \quad 0 \leq t < a \\ h(t) & \text{if} \quad t \geq a \end{cases}$$

is the same as

$$f(t) = g(t) + g(t)U(t-a) + h(t)U(t-a).$$

Similarly, a function of the type

$$f(t) = \begin{cases} 0 \quad \text{if} & 0 \leq t < a \\ g(t), \quad \text{if} & a \leq t < b \\ 0 \quad \text{if} & t \geq b \end{cases}$$

can be written

$$f(t) = g(t)\left(U(t-a) - U(t-b)\right).$$

The transform of $U(t-a)$ is

$$\begin{aligned} \mathscr{L}(U(t-a)) &= \int_0^\infty e^{-st} U(t-a)dt \\ &= \int_a^\infty e^{-st}.1dt = -\frac{e^{-st}}{s}\Big|_a^\infty \\ \mathscr{L}(U(t-a)) &= \frac{e^{-as}}{s}, \quad s > 0 \end{aligned}$$

Theorem 3.6.1 — Second shifting theorem If $F(s)$ is the Laplace transform of $f(t)$, then

$$\mathscr{L}(U_a(t)f(t-a)) = e^{-as}F(s) \tag{3.8}$$

Proof.

$$\begin{aligned} \mathscr{L}(U_a(t)f(t-a)) &= \int_0^\infty e^{-st} U_a(t) f(t-a)dt = \int_a^\infty e^{-st} f(t-a)dt \qquad \text{Let } \xi = t-a \\ &= \int_a^\infty e^{-s(\xi+a)} f(\xi)d\xi = e^{-sa}\int_a^\infty e^{-s(\xi)} f(\xi)d\xi \\ &= e^{-as}F(s) \end{aligned}$$

■

■ **Example 3.13** Let $f(t) = \begin{cases} 0, & t < 2 \\ t-2, & t \geq 2 \end{cases}$ Find $\mathscr{L}(f(t))$ ■

Solution: $U_2(t) = \begin{cases} 0, & t < 2 \\ 1, & t > 2 \end{cases} \quad \Rightarrow \quad f(t) = U_2(t)(t-2)$

$$\mathscr{L}(U_2(t)(t-2)) = e^{-2s}\mathscr{L}(t) = \frac{e^{-2s}}{s^2}$$

■ **Example 3.14** Find the inverse transform of $F(s) = \dfrac{1+e^{-2s}}{s^2}$ ■

Solution:

$$\begin{aligned} \mathscr{L}^{-1}(F(s)) &= \mathscr{L}^{-1}\left(\frac{1}{s^2} + \frac{e^{-2s}}{s^2}\right) = t + U_2(t)(t-2) \\ &= \begin{cases} t & 0 \leq t < 2 \\ 2(t-1) & t \geq 2 \end{cases} \end{aligned}$$

■ **Example 3.15** Solve the IVP

$$y'' + y = g(t), \;\; y(0) = 0, \; y'(0) = 1$$

where $g(t) = \begin{cases} 0 & 0 \le t < 1 \\ 1 & 1 \le t < 2 \end{cases}$ ■

Solution: We can express $g(t)$ as $U_1(t) - U_2(t)$.
The Laplace transform of the IVP is

$$\begin{aligned}
& \mathscr{L}(y'') + \mathscr{L}(y) = \mathscr{L}(g(t)) \\
\Rightarrow \quad & s^2\mathscr{L}(y) - sy(0) - y'(0) + \mathscr{L}(y) = \mathscr{L}(U_1(t)) - \mathscr{L}(U_2(t)) \\
\Rightarrow \quad & (s^2+1)\mathscr{L}(y(t)) - 1 = \frac{e^{-s}}{s} - \frac{e^{-2s}}{s}
\end{aligned}$$

$$\begin{aligned}
\Rightarrow \quad & \mathscr{L}(y) = \frac{1}{s^2+1} + \frac{e^{-s}}{s(s^2+1)} - \frac{e^{-2s}}{s(s^2+1)} \\
\Rightarrow \quad & y(t) = \mathscr{L}^{-1}\left(\frac{1}{s^2+1}\right) + \mathscr{L}^{-1}\left(\frac{e^{-s}}{s(s^2+1)}\right) - \mathscr{L}^{-1}\left(\frac{e^{-2s}}{s(s^2+1)}\right) \\
\Rightarrow \quad & y(t) = \sin t + U_1(t)\,[1-\cos(t-1)] - U_2(t)\,[1-\cos(t-2)]
\end{aligned}$$

3.7 Convolution

Definition 3.7.1 The convolution of the function f and g written by $f*g$ is defined by

$$(f*g)(t) = \int_0^t f(t-\tau)g(\tau)d\tau, \quad \forall t \ge 0 \tag{3.9}$$

Theorem 3.7.1 — The convolution Theorem If $F(s)$ and $G(s)$ are the Laplace transform of $f(t)$ and $g(t)$ respectively, then

$$\mathscr{L}(f*g)(t) = \mathscr{L}(f)\mathscr{L}(g) \tag{3.10}$$

Properties of convolution

1. $f*g = g*f$
2. $f*(g*h) = (f*g)*h$ (associative)
3. $f*(g+h) = (f*g)+(f*h)$ (Distributive)

■ **Example 3.16** Let $H(s) = \dfrac{1}{(s^2+\omega^2)^2}$. Find $h(t)$ ■

Solution: We have $\mathscr{L}^{-1}\left(\frac{1}{s^2+\omega^2}\right)=\frac{\sin\omega t}{\omega}$.

$$
\begin{aligned}
h(t) &= \frac{\sin\omega t}{\omega}*\frac{\sin\omega t}{\omega}=\frac{1}{\omega^2}\int_0^t \sin\omega\tau\sin\omega(t-\tau)d\tau \\
&= \frac{1}{\omega^2}\int_0^t \sin\omega\tau\sin\omega(t-\tau)d\tau \\
&= \frac{1}{\omega^2}\int_0^t \sin\omega\tau[\sin\omega t\cos\omega\tau-\sin\omega\tau\cos\omega t]d\tau \\
&= \frac{\sin\omega t}{\omega^2}\int_0^t \sin\omega\tau\cos\omega\tau-\frac{\cos\omega t}{\omega^2}\int_0^t \sin^2\omega\tau d\tau \\
&= \frac{\sin\omega t}{\omega^2}\int_0^t \sin\omega\tau\cos\omega\tau-\frac{\cos\omega t}{\omega^2}\int_0^t \left(\frac{1}{2}+\frac{\cos 2\omega\tau}{2}\right)d\tau \\
&= \frac{\sin\omega t}{\omega^2}\left[\frac{\sin^2\omega\tau}{2\omega}\right]_0^\tau-\frac{\cos\omega t}{\omega^2}\left[\frac{1}{2}t-\frac{\sin 2\omega\tau}{4\omega}\right]_0^\tau \\
&= \frac{\sin\omega t}{\omega^2}\left[\frac{\sin^2\omega t}{2\omega}\right]-\frac{t\cos\omega t}{2\omega^2}+\frac{\cos\omega t\sin 2\omega t}{4\omega^3} \\
&= \frac{\sin\omega t}{\omega^2}\left[\frac{\sin^2\omega t}{2\omega}\right]-\frac{t\cos\omega t}{2\omega^2}+\frac{\cos^2\omega t\sin\omega t}{2\omega^3} \\
&= \frac{\sin\omega t}{2\omega^3}\left[\sin^2\omega t+\cos^2\omega t\right]-\frac{t\cos\omega t}{2\omega^2} \\
&= \frac{1}{2\omega^2}\left[\frac{\sin\omega t}{\omega}-t\cos\omega t\right]
\end{aligned}
$$

■ **Example 3.17** Solve the initial value problem

$$y''+4y'+13y=2e^{-2t}\sin 3t,\quad y(0)=1,\ y'(0)=0$$

■

Solution:

Integral equation

An equation of the form

$$y(t)=f(t)+\lambda\int_0^t K(t,\tau)y(\tau)d\tau \tag{3.11}$$

is called a Volterra integral equation, where λ is a parameter and $K(t,\tau)$ is called the kernel of the integral equation. The Laplace transform is well suited to the solution of such integral equations when the kernel $K(t,\tau)$ has a special form that depends on t and τ only through the difference $t-\tau$, because then $K(t,\tau)=K(t-\tau)$ and the integral in (3.11) becomes a convolution integral.

Example 3.18 Solve the Volterra integral equation

$$y(t) = 2e^{-t} + \int_0^t \sin(t-\tau)y(\tau)d\tau$$

Solution: Taking laplace tarnsform both sides, we get

$$\begin{aligned}
\mathscr{L}(y(t)) &= \mathscr{L}\left(2e^{-t} + \int_0^t \sin(t-\tau)y(\tau)d\tau\right) \\
&= \frac{2}{1+s} + \mathscr{L}\left(\int_0^t \sin(t-\tau)y(\tau)d\tau\right) \\
&= \frac{2}{1+s} + \frac{\mathscr{L}(y(t))}{s^2+1} \\
\left(1 - \frac{1}{s^2+1}\right)\mathscr{L}(y(t)) &= \frac{2}{1+s} \Rightarrow \left(\frac{s^2}{s^2+1}\right)\mathscr{L}(y(t)) = \frac{2}{1+s} \\
\mathscr{L}(y(t)) &= \frac{2(s^2+1)}{s^2(s+1)} = \frac{2}{s^2} - \frac{2}{s} + \frac{4}{1+s} \\
y(t) &= \mathscr{L}^{-1}(\frac{2}{s^2}) - \mathscr{L}^{-1}(\frac{2}{s}) + \mathscr{L}^{-1}(\frac{4}{1+s}) \\
y(t) &= 2t - 2 + 4e^{-t}, \quad \text{for } t > 0
\end{aligned}$$

Example 3.19 Solve the equation

$$y'' + y = \int_0^t \sin(\tau)y(t-\tau)d\tau \quad y(0) = 1,\ y'(0) = 0$$

Exercise 3.6 Solve

1. $y'' + y = \sqrt{2}\sin\sqrt{2}t,\ \ y(0) = 10, y'(0) = 0$
2. $y' + y = e^{-3t}\cos 2t,\ \ y(0) = 0$
3. $y' + 2y = f(t),\ y(0) = 0,\ \ f(t) = \begin{cases} t, & 0 \le t < 1 \\ 0, & t \ge 1 \end{cases}$
4. $y' + y = \int_0^t e^{-2\tau}y(t-\tau)d\tau,\ \ y(0) = 3$
5. $y'' - y = \int_0^t \sinh(\tau)y(t-\tau)d\tau \quad y(0) = 1,\ y'(0) = 0$
6. $y'' - 4y = 2\int_0^t \sinh(2\tau)y(t-\tau)d\tau \quad y(0) = 1,\ y'(0) = 0$

3.8 Laplace Transform of the Integral of a function

Theorem 3.8.1 — Integration of $f(t)$ Let $F(s)$ be the Laplace transform of $f(t)$. If $f(t)$ is piecewise continuous and satisfies an inequality of the form $|f(t)| \le Me^{kt}$, then

$$\mathscr{L}\left(\int_0^t f(\tau)d\tau\right) = \frac{1}{s}F(s), \quad (s > 0,\ s > k) \tag{3.12}$$

Or if we take the inverse transform on both sides

$$\int_0^t f(\tau)d\tau = \mathscr{L}^{-1}\left(\frac{1}{s}F(s)\right) \tag{3.13}$$

Example 3.20 Let $\mathscr{L}(f(t)) = \dfrac{1}{s^2(s^2+\omega^2)}$. Find $f(t)$

Solution: We have $\mathscr{L}^{-1}\left(\dfrac{1}{s^2+\omega^2}\right) = \dfrac{1}{\omega}\sin\omega t$. From (3.13) it follows that

$$\begin{aligned}
\mathscr{L}^{-1}\left(\frac{1}{s(s^2+\omega^2)}\right) &= \frac{1}{\omega}\int_0^t \sin\omega\tau d\tau \\
&= \frac{1}{\omega}\left(-\frac{\cos\omega\tau}{\omega}\Big|_0^t\right) = \frac{1}{\omega}\left(\frac{-\cos\omega t+1}{\omega}\right) \\
&= \frac{1}{\omega^2}(1-\cos\omega t)
\end{aligned}$$

$$\begin{aligned}
\mathscr{L}^{-1}\left(\frac{1}{s}\left(\frac{1}{s(s^2+\omega^2)}\right)\right) &= \frac{1}{\omega^2}\int_0^t (1-\cos\omega\tau)\,d\tau \\
&= \frac{1}{\omega^2}\left(\tau-\frac{\sin\omega\tau}{\omega}\Big|_0^t\right) \\
&= \frac{1}{\omega^2}\left(t-\frac{\sin\omega t}{\omega}\right)
\end{aligned}$$

Electric Circuit

Consider the RLC Circuit below

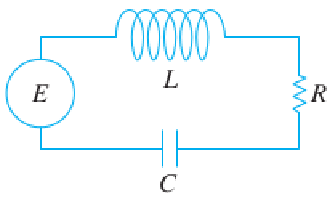


Figure 3.1: LRC series circuit.

In a single-loop or series circuit, Kirchhoff's second law states that the sum of the voltage drops across an inductor, resistor, and capacitor is equal to the impressed voltage E(t). Now it is known that the voltage drops across an inductor, resistor, and capacitor are, respectively,

$$L\frac{di(t)}{dt}\ ,Ri(\tau),\quad \text{and}\quad \frac{1}{c}\int_0^t i(\tau)d\tau$$

where $I(t)$ is the current and L, R, and C are constants. It follows that the current in a circuit, such as that shown in Figure 3.1, is governed by the **integrodifferential equation**

$$L\frac{di(t)}{dt} + Ri(\tau) + \frac{1}{c}\int_0^t i(\tau)d\tau = E(t)$$

■ **Example 3.21** Determine the current $i(t)$ in a single-loop LRC circuit when $L = 0.1\ h, R = 2\ \omega, C = 0.1\ f, i(0) = 0$, and the impressed voltage is $E(t) = 120t - 120tU(t-1)$ ■